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ORIGINAL PAPER

The use of minimum curvature surface technique in geoid computation processing of Egypt

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Abstract According to the wide spread use of satellite-based positioning techniques, especially Global Navigation Satellite Systems (GNSS), a greater attention has been paid to the precise determination of geoid models. As it is known, leveling measurements require high cost and long time in observation process that make it not convenient for the practical geodetic purposes. Thus obtaining the orthometric heights by GNSS is the most conventional way of determining these heights. Verifying this goal was the main objective behind the current research. The current research introduces a numerical solution of geoid modeling by applying a surface fitting for a few sparse data points of geoid undulation using minimum curvature surface (MCS). The MCS is presented for deriving a system of linear equations from boundary integral equations. To emphasize the precise applicability of the MCS as a tool for modeling the geoid in an area using GPS/leveling data, a comparison study between EGM2008 and MCS geoid models, is performed. The obtained results showed that MCS technique is a precise tool for determining the geoid in Egypt either on regional and/or local scale with law distortion at check points.

Keywords Geoid · Orthometric height · Ellipsoidal height · MCS · Geoid undulation · Geodetic and Cartesian coordinates · GPS

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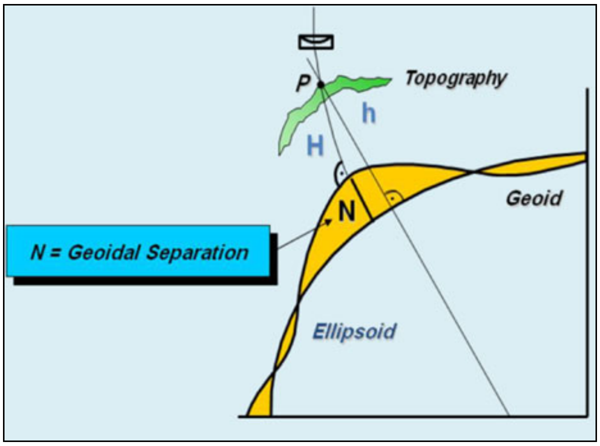
Introduction

The geoid height (or geoidal undulation) "*N*" can be defined as the separation of the reference ellipsoid with the geoid surface measured along the ellipsoidal normal as shown in the following sketch. The combined use of GPS geodetic height *"h"*, leveling orthometric height *"H"*, and geoid height "*N*" information has been a key procedure in various geodetic applications. Although these three types of height information are considerably different in terms of physical meaning, reference surface definition, observational methods, accuracy, etc., they should fulfill the simple geometrical relationship (Kotsakis and Sideris 1999): $N = h - H$

The geoid is an equipotential surface of the earth that coincides with the undisturbed mean sea level. Therefore, one might say that it describes the actual shape of the earth. The geoid is also the reference surface for most height networks since leveling gives the heights above the geoid (Harrie 1993). The geoid is determined by using several techniques based on using one or more of the different data source such as: gravimetric method using surface gravity data, satellite positioning based on measuring both ellipsoidal heights for stations with known orthometric heights, geopotential models using spherical harmonic coefficients determined from the analysis of satellite orbits, satellite altimetry using satellite-borne altimetry measurements over the ocean, astrogeodetic method using stations with measured astronomical and geodetic coordinates, and oceanographic leveling methods used mainly by the oceanographers to map the geopotential elevation of the mean surface of the ocean relative to a standard level surface (Saad and Dawood 2002). Other methods are the mathematical models similar to that used in this paper using minimum curvature surface (MCS) method.



Sketch illustrates the relation between ellipsoid height & Elevation

In this paper, precise local geoid determination will be considered according to geometric method using GPS/leveling data. First of all, an overview of the most recent Global Geoidal Model is reviewed. The mathematical approach of MCS technique is introduced. The data that are used in computing the geoid over Egypt are described as well as the two sets of data points that are used in the evaluation process of the MCS. The next section demonstrated a comparison between the results of the EGM2008 model and MCS are presented and discussed. Finally, the conclusions are drawn.

EGM2008 geoidal model

The recent release of the new Earth Gravitational Model EGM2008 by the US national Geospatial-Intelligence Agency (Pavlis et al. 2008) is undoubtedly a major breakthrough in global gravity field mapping. For the first time, a spherical harmonic model complete to degree 2190 and order 2159, is available for the Earth's external gravitational potential, for the used data sources see Fig. 1. Full access to the model's coefficients and other processing programs is available from the NGA site at: http://earthinfo.nima.mil/GandG/wgs84/gravitymod/index.html.

The EGM2008 leads to an unprecedented level of spatial sampling resolution (~9 km) for the recovery of gravity field functional contributes in a most successful way to the continuing efforts of geodetic community during the last years (and after the launch of the satellite missions CHAMP and GRACE) for a high-resolution and high accuracy reference model of Earth's static (mean) gravity field. Furthermore, it provides an indispensable tool to support new gravity field studies and other Earth monitoring projects and the ongoing development of Global Geodetic Observing System (Pavlis et al. 2008).

Following the official release of the EGM08 model, there is an expected strong interest among geodesists to quantify its actual accuracy with several validation techniques and external data sets, independently of the estimation and error calibration procedures that were used for its development. It is worthwhile to mention that the EGM2008 does not include any GPS/leveling or astronomic deflection of the vertical data.

Dawood et al. (2010) have found out that the best Global Geoidal Model that represents the gravitational field over Egypt is the EGM2008 which produced a standard deviation of undulation differences that equal to 0.23 m, which is almost identical with its global precision values. This value of constant bias of 0.23 m was taken into account. Figure 2 depicts the geoid over Egypt as calculated by EGM2008 model.

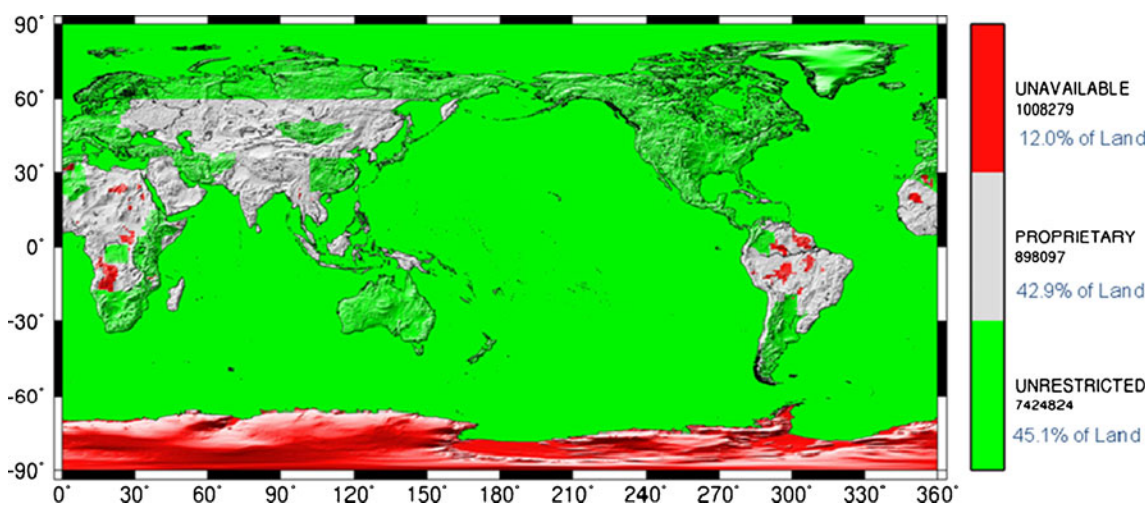


Fig. 1 A 5′×5′ Δg data availability (source: Pavlis et al. 2008)

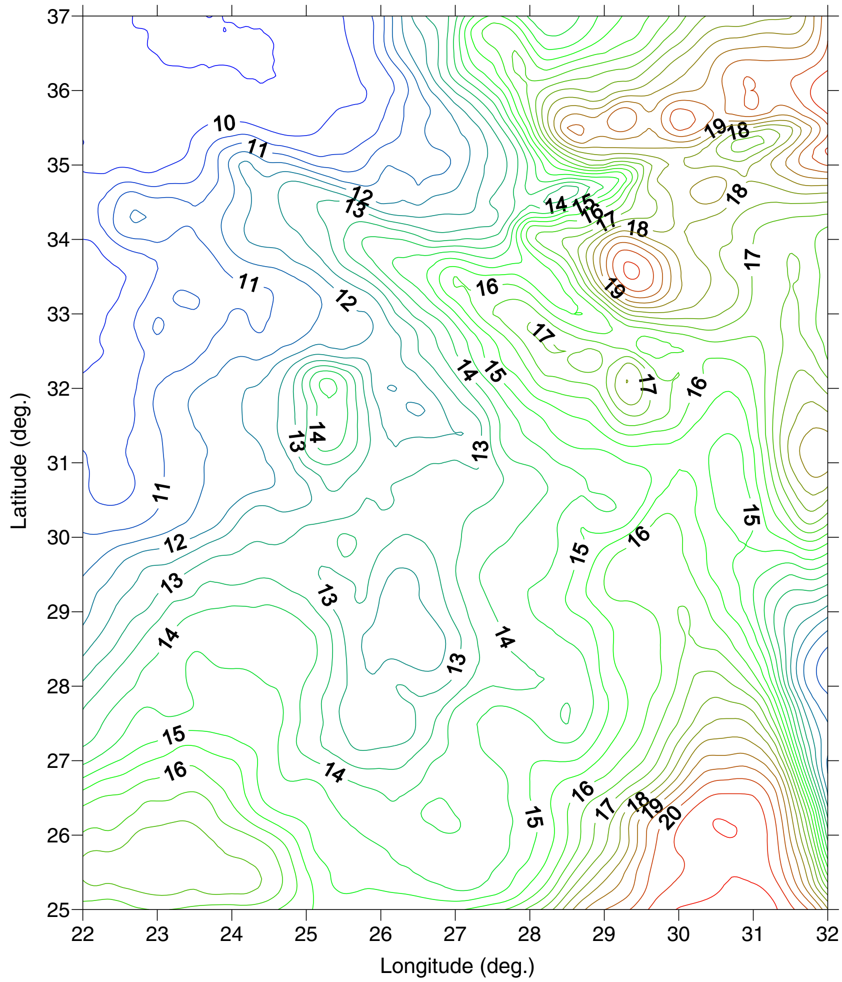


Fig. 2 The geoid undulation of Egypt as computed from EGM2008

MCS methodology

The practical methods to compute the geoid and estimating its values for little observed data available, as in Egypt, the mathematical techniques are considered in solving the related problems. Based upon the available data for Egypt, as mentioned in the report of Powell (1997), the geoid undulation in Egypt can be computed by using the mathematical techniques that are considered the best solution to compute the empirically or adjusted value of the geoid undulation. To obtain the parameters of the mathematical equations and related statistical quality indexes, the mathematical methods utilize the least square techniques to solve its mathematical equations.

According to Erol and Celik (2004), the important factors that affect the accuracy of GPS/leveling geoid model are:

– Distribution and number of reference stations (GPS/leveling stations). These points should be distributed homogeneously over the model's coverage area. In addition, they should be chosen by a way they figure out the changes of geoid surface.
– The accuracy of GPS derived ellipsoidal heights (*h*) and the heights derived from leveling measurements (*H*).
– The topographic characteristic of the geoid surface area.
– The used method in modeling the geoid.

The mathematical method of MCS is an old and over-popular approach for constructing smooth surface from irregularly spaced data. The surface of minimum curvature corresponding to the minimum of the Laplacian power or, in alternative formulation, satisfies the bi-harmonic differential equation. Physically, it models the behavior of an elastic plate. In the one-dimensional case, the minimum curvature leads to the natural cubic spline interpolation. In the two-dimensional case, a surface can be interpolated with bi-harmonic splines or gridded with an iterative finite difference scheme (EL-Shmbaky 2004).

In most practical cases, the minimum curvature technique produces a visually pleasing smooth surface. However, in case of large changes in the surface gradient, the method can create strong artificial oscillations in the unconstrained regions. Switching to lower-order methods, such minimizing the power of the gradient, solves the problem of extraneous inflections. On the other hand, it also removes the smoothness constraint and leads to gradient discontinuities (EL-Shmbaky 2004).

The mathematical formula for (MCS) is seeking for a two-dimensional surface $f(x, y)$ in region D, which is corresponding to the minimum of the Laplacian power:

$$\int\int_D \left|\nabla^2 f(x,y)\right|^2 \mathrm{d}x\mathrm{d}y \tag{1}$$

Where ∇^2 denotes the Laplacian operator $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

Alternatively, seeking $f(x, y)$ as the solution of the bi-harmonic differential equation:

$$\left(\nabla^2\right)^2 f(x,y) = 0 \tag{2}$$

Equation 1 corresponding to the normal system of equations in the least square optimization problem (Drakos 1997). On the other hand, Poisson equation can be expressed as follows:

$$(\nabla^2)^2 f(x,y) = f(x,y) \tag{3}$$

The solution of this differential equation can be solved as follows:

If $y=f(x)$ is a function of one variable, then by Taylor theorem:

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \ldots$$
$$y_3 = y_0 - hy_0' + \frac{h^2}{2!}y_0'' - \frac{h^3}{3!}y_0''' + \ldots$$

As shown in Fig. 2, by adding the two equations and neglecting the higher orders one can get $y_1 + y_3 = 2y_0 + h^2 y_0''$ with an error of less than $\left|\left(h^4 y_0''''\right)/12\right|$. $y_0'' = \frac{1}{h^2}[y_1 + y_3 - 2y_0]$ or in other format: $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{1}{h^2}[y_1 + y_3 - 2y_0]$

Similarly for a function of two variables as shown in Fig. 3:

$$\left.\begin{array}{l} \frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} = \frac{1}{h^2}[\varphi_1 + \varphi_3 - 2\varphi_0] \\ \frac{\mathrm{d}^2\varphi}{\mathrm{d}y^2} = \frac{1}{h^2}[\varphi_2 + \varphi_4 - 2\varphi_0] \end{array}\right\} \tag{4}$$

Where: φ_0 is the value of the function $f(x, y)$ at the point (x_0, y_0). It is needed to solve numerically the following partial differential equations (Fig. 4):

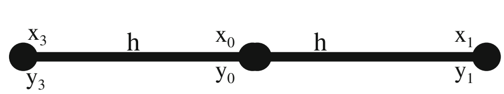


Fig. 3 The grid arms

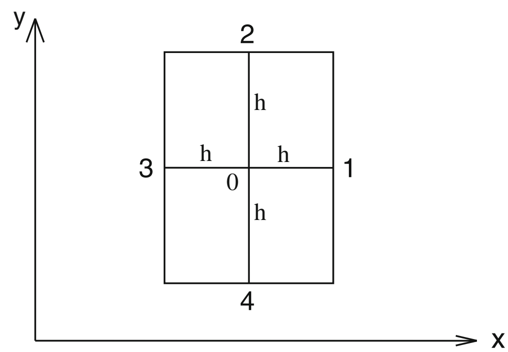


Fig. 4 The grid for two variables

1. Laplace's equation inside any closed boundary can be written as:

$$\nabla^2\varphi = 0, \text{i.e.,} \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0 \tag{5}$$

2. Poisson's equation inside any closed boundary can be written as:

$$\nabla^2\varphi = f(x,y), \text{i.e.,} \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = f(x,y) \tag{6}$$

Replacing $\frac{\partial^2\varphi}{\partial x^2}$ and $\frac{\partial^2\varphi}{\partial y^2}$ by their equivalent expression from Eqs. 5 and 6, one can get the following:

- For Laplace's equation:

$$\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 - 4\varphi_0 = 0 \tag{7}$$

- For Poisson's equation:

$$\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 - 4\varphi_0 = h^2 f(x_0, y_0) \tag{8}$$

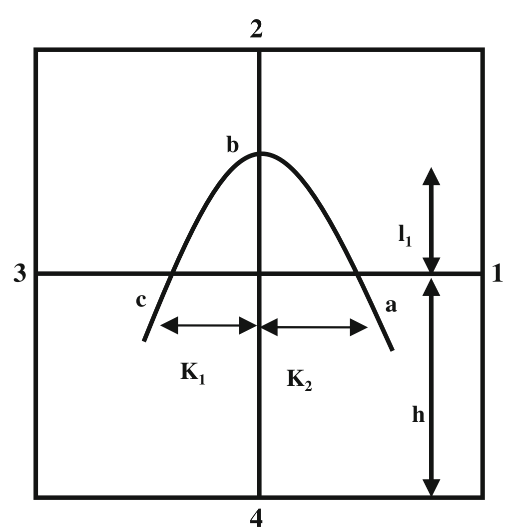


Fig. 5 Case of non-completed arms

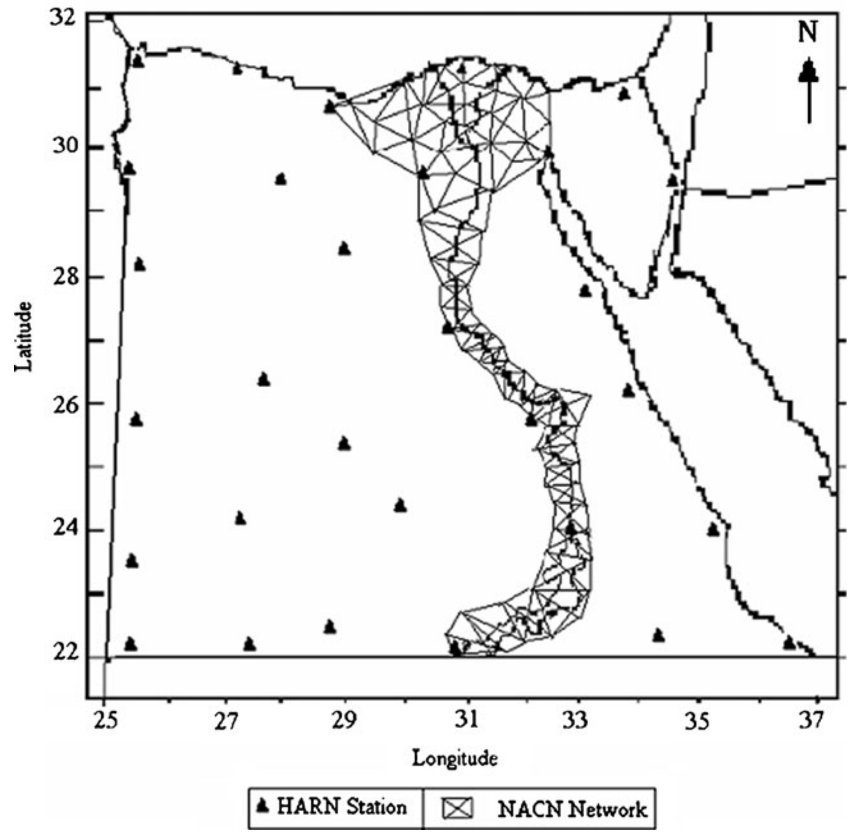


Fig. 6 The HARN and NACN networks

Fig. 7 The geometric distribution of the common points

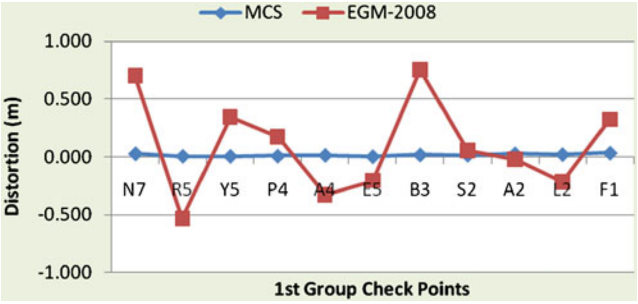


Fig. 8 The distortion at the 1st check group points as computed by EGM2008 geoidal models and MCS

The four arms about the nodes may be not completed. So, the two Eqs. 7 and 8 can be rearranged as follows:

$$\frac{2\varphi_a}{k_1(k_1+k_2)}+\frac{2\varphi_c}{k_1(k_1+k_2)}+\frac{2\varphi_b}{l_1(l_1+l_2)}+\frac{2\varphi_4}{l_1(l_1+l_2)} -\left(\frac{2}{(l_1l_2)}+\frac{2}{(k_1k_2)}\right)\phi_0=\begin{cases}0, \text{Laplac}\\ h^2f_0, \text{Poisson}\end{cases} \tag{9}$$

Where:

k_1, k_2, l_1, and l_2 are the ratio from the complete grid arm (h) shown in Fig. 5, and f_0 is a function of unknown value φ_0.

Now, the area inside the boundaries can be divided into a network or lattice of squares of side (h). The corners of these squares are called nodes of the network. The two difference Eqs. 7 and 8 are written according to the considered problem for each node. These linear equations can then be solved by least square adjustment. The parametric least square can be applied to system of equations with Laplace equation as:

$$A_{(n,m)}V_{(m,1)}=F_{(n,1)} \tag{10}$$

Where:

N The number of equations
M The number of unknown and known station
F The vector equal to observation difference
V The vector of unknown nods and no. of difference coordinates for known stations

The values of $\varphi(x, y)$ at the boundaries should be known to solve the considered problem (Sedeek 1992).

The used data

In 1995, two national GPS geodetic control networks have been established, by the Egyptian Survey Authority, to furnish a nationwide GPS skeleton for surveying and

Fig. 9 The geometric distribution of the 1st check group points (regional scale)

mapping applications. The first network is the High Accuracy Reference Network (HARN) that covers the entire Egyptian territories and consists of 30 stations with approximate separation of 200 km. The relative precision level of HARN is 1:10,000,000. The second network is the National Agricultural Cadastral Network (NACN) that is mainly covers the Nile valley and the Delta. NACN consists of 112 stations, with a station separation of 50 km approximately, whose relative precision is 1:1,000,000. Both networks are depicted in Fig. 6 (Dawood and Ismail 2005).

Unfortunately, few stations of both networks have orthometric height resulted from leveling work. Our focus only is concerned on the points that have orthometric heights, about 17 of the HARN points are taken as modeling pins (common points) with known geoid undulation. The distribution of the common points are depicted in Figs. 7 and 8. Twelve mixed stations of HARN and NACN Networks, are chosen for testing the model (the 1st group of check points) as shown in Figs. 9 and 10. The geoidal undulation values for both sets are tabulated in Tables 1 and 2, respectively. The data used for MCS evaluation process, namely the common data set and the 1st group check points are outlined in Table 3. Additionally, the computed values of EGM2008 geoidal undulation for both data sets are giv4n in both Tables 1 and 2. The differences (Distortion) between the computed values of EGM2008 geoidal undulation and the observed (GPS and orthometric heights) one are given in the last column of the two tables.

To see the contribution of the developed method in the local sense, 13 points located on the highway that connects the High dam and Aswan dam are observed by dual-frequency GPS and connected to IGS station. A precise leveling loop is connected to a first-order Bench Mark near to the High dam to determine their orthometric height. The location and the distribution of these points are presented in Fig. 11. The computed geoidal undulation values (N-obs) are given in Table 4. However, to see the influence of the developed method over the most recent Global Geoidal Models, as a reference for comparison, the geoidal undulation of all the points sets are computed from EGM2008 geoid model data.

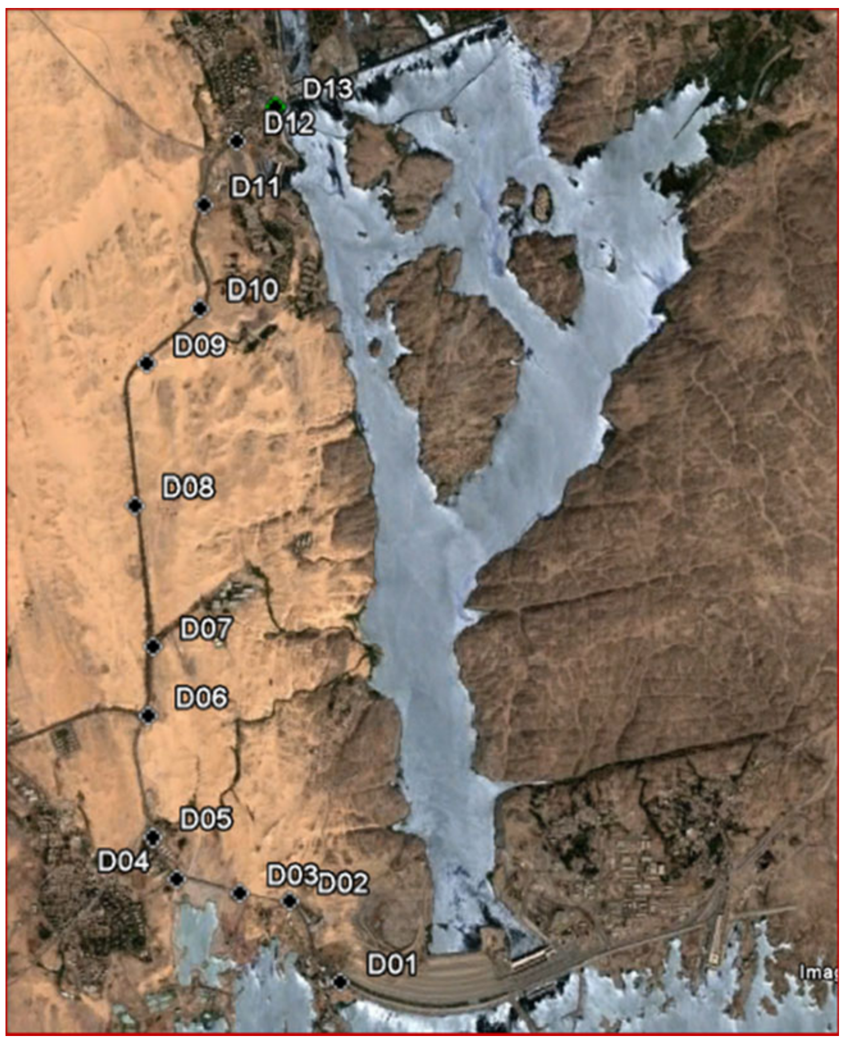


Fig. 10 The geometric distribution of the 2nd check group points (local scale)

Table 1 The location and geoidal undulation values of the common points

Old name	New name	Lat	Long	N-obs	N-EGM08	Diff.
O5	OZ02	22.422207	31.562574	9.7784	10.284	0.5056
A5	OZ07	24.041360	32.832799	11.0489	11.078	0.0291
B19	OZ08	23.940760	35.397372	10.6675	10.399	−0.2685
B20	OZ09	26.017435	34.321205	12.7134	12.698	−0.0154
M3	OZ10	25.455575	32.156737	12.1578	14.172	2.0142
I15	OZ11	25.543626	29.402926	13.172	12.888	−0.284
OZ12	OZ12	28.507226	29.096832	14.2183	15.046	0.8277
T2	OZ13	27.267525	30.779615	12.751	13.116	0.365
B11	OZ14	27.880185	33.361767	14.6447	15.987	1.3423
OZ15	OZ15	29.350026	34.772391	17.0002	15.731	−1.2692
B10	OZ16	31.119385	34.182068	17.0255	17.007	−0.0185
A6	OZ17	30.119310	32.606234	16.2067	16.146	−0.0607
OZ18	OZ18	31.595945	31.080313	17.8259	18.405	0.5791
E7	OZ19	29.834158	30.601131	14.945	15.571	0.626
D8	OZ20	30.842387	28.935306	15.0668	15.206	0.1392
X8	OZ21	31.327626	27.071952	17.2419	17.816	0.5741
Z9	OZ22	31.437845	25.398634	19.3308	20.002	0.6712

Discussions and results

MCS technique can be used as a grid transformation technique without need a priori variance covariance matrix. After constructing the grids over the area of study, the steps of solution can be summarized as in the following:

- The differences between two models are computed.
- The observation equations can be formed according to Laplace model
- Forming the reduced condition equation of the Laplace model and applying least square to give the required posteriori variance.
- The variance of the used common points is obtained and trials are stopped according to covariance of variance.
- Computing the geoid undulation at unknown grids and drawing the contour map. Calculating the distortion at the chosen points, with excluding the points with extremes values such as the points (OZ13, OZ15, OZ19, OZ20, and OZ22) as demonstrated in Table 3.

Comparisons between EGM2008 and MCS techniques in Egypt

In order to assess the performance of the MCS technique and EGM2008 model in computing the geoidal undulation over Egypt, a comparison between the resulted of MCS and EGM2008 models is presented in the following:

The geoidal undulation at the first group of check points are computed by MCS and EGM2008 models.

Table 2 The location and geoidal undulation values of the first group of check points

Old name	New name	Lat	Long	N-obs	N-EGM08	Diff.
N7	OY27	30.232176	29.840354	15.088	15.762	0.674
L5	OY35	22.752192	31.848742	11.156	10.615	−0.541
R5	OY36	22.107805	31.552215	9.974	10.31	0.336
Y5	OY41	22.206004	31.554883	10.092	10.256	0.164
P4	OZ32	24.155120	32.968164	11.397	11.054	−0.343
A4	OZ44	25.648151	32.693353	12.02	11.81	−0.21
E5	OZ52	23.429444	32.826730	10.098	10.836	0.738
B3	OZ66	27.325233	31.188597	12.842	12.883	0.041
S2	OZ68	27.411941	30.543393	13.488	13.436	−0.052
A2	OZ70	29.018091	31.159649	15.216	14.977	−0.239
L2	OZ74	28.184611	30.804617	13.738	14.028	0.29
F1	OZ97	30.028739	31.277675	15.268	15.414	0.146

Table 3 The distortion at common and 1st check points groups by MCS

Point	Distortion (m)
Common points	
O5	0.077
A5	0.003
B19	−0.229
B20	0.053
M3	−0.603
I15	0.773
OZ12	−0.702
T2	−1.079
B11	0.353
OZ15	1.493
B10	0.006
A6	−0.003
OZ18	0.329
E7	−1.068
D8	−1.839
X8	−0.118
Z9	1.084
1st group check points	
N7	0.031
R5	0.009
Y5	0.008
P4	0.010
A4	0.017
E5	0.005
B3	0.020
S2	0.017
A2	0.029
L2	0.023
F1	0.036
Max. value	0.036
Min. value	0.005
Average	0.019
SD	0.010

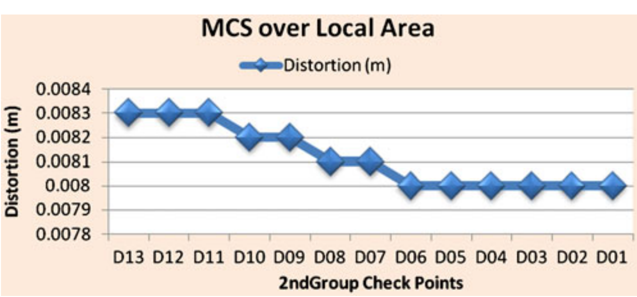


Fig. 11 The distortion at the 2nd check group points as computed by MCS

Table 4 The distortion as computed by EGM2008 and MCS

Point	EGM2008	MCS
N7	0.674	0.031
R5	−0.541	0.009
Y5	0.336	0.008
P4	0.164	0.010
A4	−0.343	0.017
E5	−0.210	0.005
B3	0.738	0.020
S2	0.041	0.017
A2	−0.052	0.029
L2	−0.239	0.023
F1	0.290	0.036
max	+0.738	0.036
min	−0.541	0.005
S.D.	0.409	0.010

The value of distortion of both models over the observed values (the true values) is computed by finding the differences between the computed and observed geoidal undulation. The value of distortion of both models, maximum and minimum distortion and the standard deviation of distortion are shown in Table 4. The value of distortion of both models are depicted in Figs. 7 and 8.

As indicated in Table 4 and Figs. 7 and 8, it is obvious that the MCS technique gives the best values of distortion

Table 5 The distortion at 2nd check points group by MCS

Point	Distortion (m)
2nd group check points	
D13	0.0083
D12	0.0083
D11	0.0083
D10	0.0082
D09	0.0082
D08	0.0081
D07	0.0081
D06	0.0080
D05	0.0080
D04	0.0080
D03	0.0080
D02	0.0080
D01	0.0080
Max. ve	0.0083
Min. ve	0.0080
Average	0.0081
STDEV	0.0001

(minimum values) over Egypt where the computed distortions are ranged between 5 mm to 3.6 cm while EGM2008 distortion values ranged between 73.8 and 54.1 cm. Hence, it is easy to see that the geoid undulation computed by MCS has the highest accuracy.

To see the performance of the MCS technique in computing the geoidal undulation over a local area, the second group of check points was utilized. The geoidal undulation of the thirteen points of the check set is computed by the MCS. The distortion values of checked points over the observed geoidal undulation values (the true values) can be found by computing the differences between the computed and observed values. The resulted value of distortion, maximum and minimum distortion and the standard deviation of distortion are shown in Table 5. The value of distortion of both models are illustrated in Figs. 9 and 10.

As it is shown in Table 5 and Figs. 9 and 10, the differences in the geoid undulation of the 2nd group check points are varied between 8.3 and 8 mm. The results of the 2nd group confirm again the precise applicability of MCS technique in computing the geoidal undulation over local areas.

Conclusion

Based on the previous analysis and the obtained numerical results, the following conclusions can be drawn:

- Among available data and techniques, GPS/Leveling with MCS technique might be the most appropriate combination for geoid model precise outputs in Egypt.
- The MCS technique gives a best geoid undulation over the EGM2008 geoid models in Egypt and it is recommend to use MCS technique to compute the geoid undulation in Egypt.

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